

# ELC 4351: Digital Signal Processing

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# Frequency Analysis of Signals

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# The Fourier Series for Continuous-Time Periodic Signals

A linear combination of harmonics (harmonically related complex exponentials):

## Synthesis Equation

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi k F_0 t}$$

## Analysis Equation

$$c_k = \frac{1}{T_p} \int_{T_p} x(t) e^{-j2\pi k F_0 t} dt$$

where, the fundamental period is  $T_p = 1/F_0$ .

# The Fourier Series for Continuous-Time Periodic Signals

A linear combination of cosine functions, if signal  $x(t)$  is real:

## Synthesis Equation

$$x(t) = a_0 + \sum_{k=1}^{\infty} (a_k \cos 2\pi k F_0 t - b_k \sin 2\pi k F_0 t)$$

where

$$a_0 = c_0$$

$$a_k = 2|c_k| \cos \theta_k$$

$$b_k = 2|c_k| \sin \theta_k$$

$$c_k = |c_k| e^{j\theta_k}$$

# The Fourier Series for Continuous-Time Periodic Signals

The Dirichlet conditions guarantee that  $x(t)$  and its Fourier series representation are equal at any value of  $t$ :

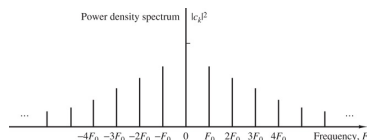
- 1  $x(t)$  has a finite number of discontinuities in any period.
- 2  $x(t)$  contains a finite number of maxima and minima during any period.
- 3  $x(t)$  is absolutely integrable in any period, i.e.,  $\int_{T_p} |x(t)| dt < \infty$ .

# Power Density Spectrum of Periodic Signals

A periodic signal has a finite average power

$$\begin{aligned}P_x &= \frac{1}{T_p} \int_{T_p} |x(t)|^2 dt \\&= \frac{1}{T_p} \int_{T_p} x(t)x^*(t) dt \\&= \frac{1}{T_p} \int_{T_p} x(t) \sum_{k=-\infty}^{\infty} c_k^* e^{-j2\pi kF_0 t} dt \\&= \sum_{k=-\infty}^{\infty} c_k^* \left[ \frac{1}{T_p} \int_{T_p} x(t) e^{-j2\pi kF_0 t} dt \right] \\&= \sum_{k=-\infty}^{\infty} |c_k|^2 \quad (\text{Parseval's Relation})\end{aligned}$$

$$P_x = a_0^2 + \frac{1}{2} \sum_{k=1}^{\infty} (a_k^2 + b_k^2)$$



# The Fourier Transform for Continuous-Time Aperiodic Signals

Going from periodic signal to aperiodic signal, we make the period  $T_p \rightarrow \infty$ .

$$x(t) = \lim_{T_p \rightarrow \infty} x_p(t)$$

$$x_p(t) = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi k F_0 t}, \quad F_0 = 1/T_p$$

$$\begin{aligned} c_k &= \frac{1}{T_p} \int_{-T_p/2}^{T_p/2} x(t) e^{-j2\pi k F_0 t} dt \\ &= \frac{1}{T_p} \underbrace{\int_{-\infty}^{\infty} x(t) e^{-j2\pi k F_0 t} dt}_{X(F)} \end{aligned}$$

# The Fourier Transform for Continuous-Time Aperiodic Signals

We write  $F \triangleq kF_0 = k/T_p$  and  $\Delta F \triangleq F_0 = 1/T_p$ .  
As  $T_p \rightarrow \infty$ ,  $\Delta F = dF$ . Therefore

$$\begin{aligned}x_p(t) &= \frac{1}{T_p} \sum_{k=-\infty}^{\infty} X(F) e^{j2\pi kF_0 t} \\&= \sum_{k=-\infty}^{\infty} X(k\Delta F) e^{j2\pi kF_0 t} \Delta F \\x(t) &= \lim_{T_p \rightarrow \infty} x_p(t) \\&= \lim_{\Delta F \rightarrow 0} \sum_{k=-\infty}^{\infty} X(k\Delta F) e^{j2\pi kF_0 t} \Delta F \\&= \int_{-\infty}^{\infty} X(F) e^{j2\pi Ft} dF\end{aligned}$$



# The Fourier Transform for Continuous-Time Aperiodic Signals

## Synthesis Equation (Inverse Transform)

$$x(t) = \int_{-\infty}^{\infty} X(F) e^{j2\pi Ft} dF$$

## Analysis Equation (Direct Transform)

$$X(F) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi Ft} dt$$

# Energy Density Spectrum of Aperiodic Signals

Signal Energy:  $E_x = \int_{-\infty}^{\infty} |x(t)|^2 dt$

$$\begin{aligned} E_x &= \int_{-\infty}^{\infty} x(t)x^*(t) dt \\ &= \int_{-\infty}^{\infty} x(t) dt \left[ \int_{-\infty}^{\infty} X^*(F)e^{-j2\pi Ft} dF \right] \\ &= \int_{-\infty}^{\infty} X^*(F)dF \left[ \int_{-\infty}^{\infty} x(t)e^{-j2\pi Ft} dt \right] \\ &= \int_{-\infty}^{\infty} X^*(F)X(F)dF \\ &= \int_{-\infty}^{\infty} |X(F)|^2 dF \end{aligned}$$

## Parseval's Relation

$$E_x = \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} |X(F)|^2 dF$$

# Energy Density Spectrum of Aperiodic Signals

Energy Density Spectrum:

$$S_{xx}(F) \triangleq |X(F)|^2$$

Therefore,  $S_{xx}(F) \geq 0$ , for all  $F$ .

If signal  $x(t)$  is real,  $|X(-F)| = |X(F)|$  and  $\angle X(-F) = -\angle X(F)$ . It follows that

$$S_{xx}(-F) = S_{xx}(F)$$

# The Fourier Series of Discrete-Time Periodic Signals

$x(n)$  is periodic with period  $N$ . That is,  $x(n) = x(n + N)$  for all  $n$ .

A linear combination of  $N$  harmonically related exponents:

## Synthesis Equation

$$x(n) = \sum_{k=0}^{N-1} c_k e^{j2\pi kn/N}$$

## Analysis Equation

$$c_k = \frac{1}{N} \sum_{n=0}^{N-1} x(n) e^{-j2\pi kn/N}$$

# The Fourier Series of Discrete-Time Periodic Signals

The Fourier series coefficients  $\{c_k\}$  is a periodic sequence with fundamental period  $N$  (when extended outside the range  $[0, N - 1]$ ).

$$\begin{aligned}c_{k+N} &= \frac{1}{N} \sum_{n=0}^{N-1} x(n) e^{-j2\pi(k+N)n/N} \\ &= \frac{1}{N} \sum_{n=0}^{N-1} x(n) e^{-j2\pi kn/N} \\ &= c_k\end{aligned}$$

The spectrum of  $x(n)$  is a periodic sequence with period  $N$ .

# The Fourier Series of Discrete-Time Periodic Signals

A linear combination of cosine functions, if signal  $x(n)$  is real:

## Synthesis Equation

$$x(n) = a_0 + 2 \sum_{k=1}^L (a_k \cos(2\pi kn/N) - b_k \sin(2\pi kn/N))$$

where

$$a_0 = c_0$$

$$a_k = 2|c_k| \cos \theta_k$$

$$b_k = 2|c_k| \sin \theta_k$$

$$L = \begin{cases} N/2 & \text{if } N \text{ is even} \\ (N-1)/2 & \text{if } N \text{ is odd} \end{cases}$$

# Power Density Spectrum of Periodic Signals

The average power of a discrete-time periodic signal with period  $N$ :

$$\begin{aligned} P_x &= \frac{1}{N} \sum_{n=0}^{N-1} |x(n)|^2 \\ &= \frac{1}{N} \sum_{n=0}^{N-1} x(n)x^*(n) \\ &= \frac{1}{N} \sum_{n=0}^{N-1} x(n) \left( \sum_{k=0}^{N-1} c_k^* e^{-j2\pi kn/N} \right) \\ &= \sum_{k=0}^{N-1} c_k^* \left[ \frac{1}{N} \sum_{n=0}^{N-1} x(n) e^{-j2\pi kn/N} \right] \\ &= \sum_{k=0}^{N-1} |c_k|^2 \end{aligned}$$



# Power Density Spectrum of Periodic Signals

Energy over a signal period:

$$E_N = \sum_{n=0}^{N-1} |x(n)|^2 = N \sum_{k=0}^{N-1} |c_k|^2$$

If  $x(n)$  is real,  $c_k^* = c_{-k}$ . Equivalently,  $|c_{-k}| = |c_k|$  and  $-\angle c_{-k} = \angle c_k$ .

# The Fourier Transform of Discrete-Time Aperiodic Signals

## Analysis Equation

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n}, \quad \omega \in [-\pi, \pi) \text{ or } \omega \in [0, 2\pi)$$

## Synthesis Equation

$$x(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\omega)e^{j\omega n} d\omega$$

$X(\omega)$  is periodic with period  $2\pi$ :

$$\begin{aligned} X(\omega + 2\pi k) &= \sum_{n=-\infty}^{\infty} x(n)e^{-j(\omega+2\pi k)n} \\ &= \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n} = X(\omega) \end{aligned}$$

# Convergence of the Fourier Transform

$$X_N(\omega) = \sum_{n=-N}^N x(n)e^{-j\omega n}$$

Uniform convergence:

$$\lim_{N \rightarrow \infty} \left\{ \sup_{\omega} |X(\omega) - X_N(\omega)| \right\} = 0, \quad \text{for all } \omega$$

Uniform convergence is guaranteed if  $\sum_{n=-\infty}^{\infty} |x(n)| < \infty$ .

Mean-square convergence:

$$\lim_{N \rightarrow \infty} \int_{-\pi}^{\pi} |X(\omega) - X_N(\omega)|^2 d\omega = 0, \quad \text{for all } \omega$$

Mean-square convergence is for finite-energy signals  $\sum_{n=-\infty}^{\infty} |x(n)|^2 < \infty$ .

# Energy Density Spectrum of Aperiodic Signals

The energy of a discrete-time signal  $x(n)$ :

$$\begin{aligned} E_x &= \sum_{n=-\infty}^{\infty} |x(n)|^2 \\ &= \sum_{n=-\infty}^{\infty} x(n)x^*(n) \\ &= \sum_{n=-\infty}^{\infty} x(n) \left[ \frac{1}{2\pi} \int_{-\pi}^{\pi} X^*(\omega) e^{-j\omega n} d\omega \right] \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} X^*(\omega) \left[ \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n} \right] d\omega \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} |X(\omega)|^2 d\omega \end{aligned}$$

# Energy Density Spectrum of Aperiodic Signals

Energy Density Spectrum:

$$S_{xx}(\omega) \triangleq |X(\omega)|^2$$

If  $x(n)$  is real,  $X^*(\omega) = X(-\omega)$ . Equivalently,  $|X(-\omega)| = |X(\omega)|$  and  $\angle X(-\omega) = -\angle X(\omega)$ . It follows that

$$S_{xx}(-\omega) = S_{xx}(\omega)$$

# Relationship of the Fourier Transform to the z-Transform

## z-Transform

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)z^{-n}; \quad \text{ROC: } r_2 < |z| < r_1$$

$z$  in polar form:  $z = re^{j\omega}$ . We have

$$X(z) = \sum_{n=-\infty}^{\infty} [x(n)r^{-n}]e^{-j\omega n}$$

If  $X(z)$  converges for  $|z| = 1$ ,

$$X(z) \big|_{z=e^{j\omega}} = X(\omega) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n}$$

# Relationship of the Fourier Transform to the z-Transform

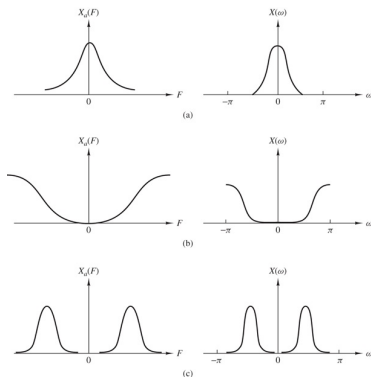
$$X(z) \Big|_{z=e^{j\omega}} = X(\omega) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n}$$

Fourier transform can be viewed as the z-transform of the sequence evaluated on the unit circle.

If  $X(z)$  does not converge in the region  $|z| = 1$ , the Fourier transform  $X(\omega)$  does not exist.

# Frequency-Domain Classification of Signals: The Concept of Bandwidth

Power (energy) density spectrum  
concentration { low-frequency  
high-frequency  
bandpass



Bandwidth — a quantitative measure

Suppose a continuous-time signal has 90% of its power (energy) density spectrum in range  $F_1 < F < F_2$ . The 90% bandwidth of the signal is  $F_2 - F_1$ .



# Frequency-Domain Classification of Signals: The Concept of Bandwidth

Narrowband:  $F_2 - F_1 \ll \frac{F_1 + F_2}{2}$  (median frequency)

Wideband: Otherwise

Bandlimited:  $X(F) = 0$  for  $|F| > B$   
 $X(\omega) = 0$  for  $\omega_0 < |\omega| < \pi$

No signal can be time-limited and band-limited simultaneously.  
(Reciprocal relationship)