ELC 4351: Digital Signal Processing

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Frequency Analysis of Signals

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The Fourier Series for Continuous-Time Periodic Signals

A linear combination of harmonics (harmonically related complex exponentials):

Synthesis Equation

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi kF_0 t}$$

Analysis Equation

$$c_k = \frac{1}{T_p} \int_{T_p} x(t) e^{-j2\pi kF_0 t} dt$$

where, the fundamental period is $T_p = 1/F_0$.

The Fourier Series for Continuous-Time Periodic Signals

A linear combination of cosine functions, if signal $x(t)$ is real:

Synthesis Equation

$$x(t) = a_0 + \sum_{k=1}^{\infty}(a_k cos2\pi kF_0t - b_k sin2\pi kF_0t)$$

where

$$\begin{aligned} a_0 &= c_0 \\ a_k &= 2|c_k|cos\theta_k \\ b_k &= 2|c_k|sin\theta_k \\ c_k &= |c_k|e^{j\theta_k} \end{aligned}$$

The Fourier Series for Continuous-Time Periodic Signals

The Dirichlet conditions guarantee that $x(t)$ and its Fourier series representation are equal at any value of t:

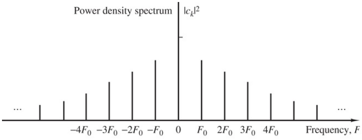
1. $x(t)$ has a finite number of discontinuities in any period.
2. $x(t)$ contains a finite number of maxima and minima during any period.
3. $x(t)$ is absolutely integrable in any period, i.e. $\int_{T_p} |x(t)|dt < \infty$.

Power Density Spectrum of Periodic Signals

A periodic signal has a finite average power

$$
\begin{aligned}
P_x &= \frac{1}{T_p}\int_{T_p} |x(t)|^2 dt \\
&= \frac{1}{T_p}\int_{T_p} x(t)x^*(t)dt \\
&= \frac{1}{T_p}\int_{T_p} x(t)\sum_{k=-\infty}^{\infty} c_k^* e^{-j2\pi kF_0 t}dt \\
&= \sum_{k=-\infty}^{\infty} c_k^* \left[\frac{1}{T_p}\int_{T_p} x(t)e^{-j2\pi kF_0 t}dt\right] \\
&= \sum_{k=-\infty}^{\infty} |c_k|^2 \qquad \text{(Parseval's Relation)}
\end{aligned}
$$

$$P_x = a_0^2 + \frac{1}{2}\sum_{k=1}^{\infty}(a_k^2 + b_k^2)$$



The Fourier Transform for Continuous-Time Aperiodic Signals

Going from periodic signal to aperiodic signal, we make the period $T_p \to \infty$.

$$
\begin{aligned}
x(t) &= \lim_{T_p \to \infty} x_p(t) \\
x_p(t) &= \sum_{k=-\infty}^{\infty} c_k e^{j2\pi k F_0 t}, \qquad F_0 = 1/T_p \\
c_k &= \frac{1}{T_p} \int_{-T_p/2}^{T_p/2} x(t) e^{-j2\pi k F_0 t} dt \\
&= \frac{1}{T_p} \underbrace{\int_{-\infty}^{\infty} x(t) e^{-j2\pi k F_0 t} dt}_{X(F)}
\end{aligned}
$$

The Fourier Transform for Continuous-Time Aperiodic Signals

We write $F \triangleq kF_0 = k/T_p$ and $\Delta F \triangleq F_0 = 1/T_p$.
As $T_p \to \infty$, $\Delta F = dF$. Therefore

$$
\begin{aligned}
x_p(t) &= \frac{1}{T_p} \sum_{k=-\infty}^{\infty} X(F) e^{j2\pi k F_0 t} \\
&= \sum_{k=-\infty}^{\infty} X(k\Delta F) e^{j2\pi k F_0 t} \Delta F \\
x(t) &= \lim_{T_p \to \infty} x_p(t) \\
&= \lim_{\Delta F \to 0} \sum_{k=-\infty}^{\infty} X(k\Delta F) e^{j2\pi k F_0 t} \Delta F \\
&= \int_{-\infty}^{\infty} X(F) e^{j2\pi F t} dF
\end{aligned}
$$

The Fourier Transform for Continuous-Time Aperiodic Signals

Synthesis Equation (Inverse Transform)

$$x(t) = \int_{-\infty}^{\infty} X(F) e^{j2\pi Ft} dF$$

Analysis Equation (Direct Transform)

$$X(F) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi Ft} dt$$

Energy Density Spectrum of Aperiodic Signals

Signal Energy: $E_x = \int_{-\infty}^{\infty} |x(t)|^2 dt$

$$
\begin{aligned}
E_x &= \int_{-\infty}^{\infty} x(t)x^*(t)dt \\
&= \int_{-\infty}^{\infty} x(t)dt \left[\int_{-\infty}^{\infty} X^*(F)e^{-j2\pi Ft}dF\right] \\
&= \int_{-\infty}^{\infty} X^*(F)dF \left[\int_{-\infty}^{\infty} x(t)e^{-j2\pi Ft}dt\right] \\
&= \int_{-\infty}^{\infty} X^*(F)X(F)dF \\
&= \int_{-\infty}^{\infty} |X(F)|^2 dF
\end{aligned}
$$

Energy Density Spectrum of Aperiodic Signals

Parseval's Relation

$$E_x = \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} |X(F)|^2 dF$$

Energy Density Spectrum of Aperiodic Signals

Energy Density Spectrum:

$$S_{xx}(F) \triangleq |X(F)|^2$$

Therefore, $S_{xx}(F) \geq 0$, for all F.

If signal $x(t)$ is real, $|X(-F)| = |X(F)|$ and $\angle X(-F) = -\angle X(F)$. It follows that

$$S_{xx}(-F) = S_{xx}(F)$$

The Fourier Series of Discrete-Time Periodic Signals

$x(n)$ is periodic with period N. That is, $x(n) = x(n+N)$ for all n.

A linear combination of N harmonically related exponents:

Synthesis Equation

$$x(n) = \sum_{k=0}^{N-1} c_k e^{j2\pi kn/N}$$

Analysis Equation

$$c_k = \frac{1}{N} \sum_{n=0}^{N-1} x(n) e^{-j2\pi kn/N}$$

The Fourier Series of Discrete-Time Periodic Signals

The Fourier series coefficients $\{c_k\}$ is a periodic sequence with fundamental period N (when extended outside the range $[0, N-1]$).

$$
\begin{aligned}
c_{k+N} &= \frac{1}{N}\sum_{n=0}^{N-1} x(n)e^{-j2\pi(k+N)n/N} \\
&= \frac{1}{N}\sum_{n=0}^{N-1} x(n)e^{-j2\pi kn/N} \\
&= c_k
\end{aligned}
$$

The spectrum of $x(n)$ is a periodic sequence with period N.

The Fourier Series of Discrete-Time Periodic Signals

A linear combination of cousin functions, if signal $x(n)$ is real:

Synthesis Equation

$$x(n) = a_0 + 2\sum_{k=1}^{L} \left(a_k \cos(2\pi kn/N) - b_k \sin(2\pi kn/N)\right)$$

where

$$\begin{aligned} a_0 &= c_0 \\ a_k &= 2|c_k|\cos\theta_k \\ b_k &= 2|c_k|\sin\theta_k \\ L &= \begin{cases} N/2 & \text{if } N \text{ is even} \\ (N-1)/2 & \text{if } N \text{ is odd} \end{cases} \end{aligned}$$

Power Density Spectrum of Periodic Signals

The average power of a discrete-time periodic signal with period N:

$$
\begin{aligned}
P_x &= \frac{1}{N}\sum_{n=0}^{N-1}|x(n)|^2 \\
&= \frac{1}{N}\sum_{n=0}^{N-1}x(n)x^*(n) \\
&= \frac{1}{N}\sum_{n=0}^{N-1}x(n)\left(\sum_{k=0}^{N-1}c_k^* e^{-j2\pi kn/N}\right) \\
&= \sum_{k=0}^{N-1}c_k^*\left[\frac{1}{N}\sum_{n=0}^{N-1}x(n)e^{-j2\pi kn/N}\right] \\
&= \sum_{k=0}^{N-1}|c_k|^2
\end{aligned}
$$

Power Density Spectrum of Periodic Signals

Energy over a signal period:

$$E_N = \sum_{n=0}^{N-1} |x(n)|^2 = N \sum_{k=0}^{N-1} |c_k|^2$$

If $x(n)$ is real, $c_k^* = c_{-k}$. Equivalently, $|c_{-k}| = |c_k|$ and $-\angle c_{-k} = \angle c_k$.

The Fourier Transform of Discrete-Time Aperiodic Signals

Analysis Equation

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}, \qquad \omega \in [-\pi, \pi) \text{ or } \omega \in [0, 2\pi)$$

Synthesis Equation

$$x(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\omega) e^{j\omega n} d\omega$$

$X(\omega)$ is periodic with period 2π:

$$\begin{aligned} X(\omega + 2\pi k) &= \sum_{n=-\infty}^{\infty} x(n) e^{-j(\omega+2\pi k)n} \\ &= \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n} = X(\omega) \end{aligned}$$

Convergence of the Fourier Transform

$$X_N(\omega) = \sum_{n=-N}^{N} x(n)e^{-j\omega n}$$

Uniform convergence:

$$\lim_{N\to\infty} \{\sup_{\omega} |X(\omega) - X_N(\omega)|\} = 0, \quad \text{for all } \omega$$

Uniform convergence is guaranteed if $\sum_{n=-\infty}^{\infty} |x(n)| < \infty$.

Mean-square convergence:

$$\lim_{N\to\infty} \int_{-\pi}^{\pi} |X(\omega) - X_N(\omega)|^2 d\omega = 0, \quad \text{for all } \omega$$

Mean-square convergence is for finite-energy signals $\sum_{n=-\infty}^{\infty} |x(n)|^2 < \infty$.

Energy Density Spectrum of Aperiodic Signals

The energy of a discrete-time signal $x(n)$:

$$\begin{aligned}
E_x &= \sum_{n=-\infty}^{\infty} |x(n)|^2 \\
&= \sum_{n=-\infty}^{\infty} x(n)x^*(n) \\
&= \sum_{n=-\infty}^{\infty} x(n)\left[\frac{1}{2\pi}\int_{-\pi}^{\pi} X^*(\omega)e^{-j\omega n}d\omega\right] \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} X^*(\omega)\left[\sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n}\right]d\omega \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} |X(\omega)|^2 d\omega
\end{aligned}$$

Energy Density Spectrum of Aperiodic Signals

Energy Density Spectrum:

$$S_{xx}(\omega) \triangleq |X(\omega)|^2$$

If $x(n)$ is real, $X^*(\omega) = X(-\omega)$. Equivalently, $|X(-\omega)| = |X(\omega)|$ and $\angle X(-\omega) = -\angle X(\omega)$. It follows that

$$S_{xx}(-\omega) = S_{xx}(\omega)$$

Relationship of the Fourier Transform to the z-Transform

z-Transform

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)z^{-n}; \qquad \text{ROC: } r_2 < |z| < r_1$$

z in polar form: $z = re^{j\omega}$. We have

$$X(z) = \sum_{n=-\infty}^{\infty} [x(n)r^{-n}]e^{-j\omega n}$$

If $X(z)$ converges for $|z| = 1$,

$$X(z)\,|_{z=e^{j\omega}} = X(\omega) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n}$$

Relationship of the Fourier Transform to the z-Transform

$$X(z)\,|_{z=e^{j\omega}} = X(\omega) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n}$$

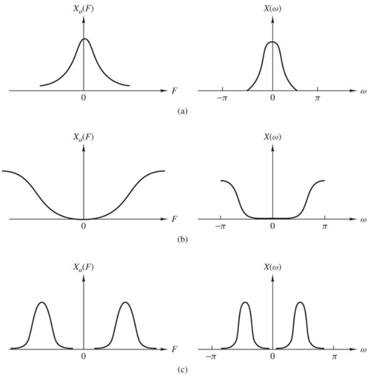
Fourier transform can be viewed as the z-transform of the sequence evaluated on the unit circle.

If $X(z)$ does not converge in the region $|z| = 1$, the Fourier transform $X(\omega)$ does not exist.

Frequency-Domain Classification of Signals: The Concept of Bandwidth

Power (energy) density spectrum

concentration
- low-frequency
- high-frequency
- bandpass



Bandwidth — a quantitative measure

Suppose a continuous-time signal has 90% of its power (energy) density spectrum in range $F_1 < F < F_2$. The 90% bandwidth of the signal is $F_2 - F_1$.

Frequency-Domain Classification of Signals: The Concept of Bandwidth

Narrowband: $F_2 - F_1 \ll \frac{F_1+F_2}{2}$ (median frequency)
Wideband: Otherwise

Bandlimited:
$$X(F) = 0 \quad \text{for } |F| > B$$
$$X(\omega) = 0 \quad \text{for } \omega_0 < |\omega| < \pi$$

No signal can be time-limited and band-limited simultaneously. (Reciprocal relationship)