ELC 4351: Digital Signal Processing

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Frequency-Domain Analysis of LTI Systems III

Frequency-domain Analysis of LTI Systems

Inverse Systems and Deconvolution

Minimum-Phase, Maximum-Phase, and Mixed-Phase Systems

System Identification and Deconvolution

Inverse Systems and Deconvolution

- In many practical applications we are given an output signal from a system whose characteristics are unknown and we are asked to determine the input signal.
- Channel distortion and a need for a corrective system: Equalizer, Inverse system
- An inverse system — The corrective system has a frequency response which is basically the reciprocal of the frequency response of the system that caused the distortion.
- Deconvolution — The inverse system operation that takes $y(n)$ and produces $x(n)$.
- System Identification — In short, to find $h(n)$ or $H(\omega)$.

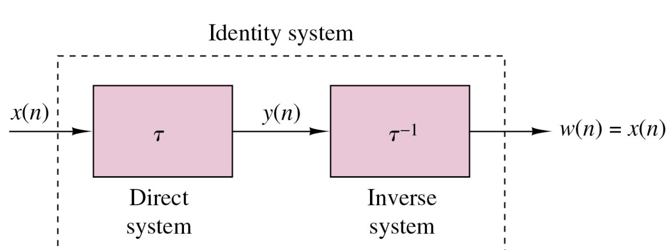
Invertibility of Linear Time-Invariant Systems

A system is said to be *invertible* if there is a one-to-one correspondence between its input and output signals.

An invertible system: $\mathcal{T}$

The inverse system: $\mathcal{T}^{-1}$

$$w(n) = \mathcal{T}^{-1}[y(n)] = \mathcal{T}^{-1}\{\mathcal{T}[x(n)]\} = x(n)$$



Invertibility of Linear Time-Invariant Systems

LTI system $\mathcal{T}$ has impulse response $h(n)$; the inverse system $\mathcal{T}^{-1}$ has impulse response $h_I(n)$.

$$w(n) = h_I(n) \otimes h(n) \otimes x(n) = x(n)$$

$$h(n) \otimes h_I(n) = \delta(n)$$

$$H(z)H_I(z) = 1$$

Therefore,

$$H_I(z) = \frac{1}{H(z)}$$

Invertibility of Linear Time-Invariant Systems

LTI system $\mathcal{T}$ has impulse response $h(n)$; the inverse system $\mathcal{T}^{-1}$ has impulse response $h_I(n)$.

$$H_I(z) = \frac{1}{H(z)}$$

If $H(z)$ has a rational system function

$$H(z) = \frac{B(z)}{A(z)}$$

then

$$H_I(z) = \frac{A(z)}{B(z)}$$

- The zeros of $H(z)$ become the poles of the inverse system, and vice versa.
- If $H(z)$ is an FIR system, then $H_I(z)$ is an all-pole system, and vice versa.

Invertibility of Linear Time-Invariant Systems

$$h(n) \otimes h_I(n) = \delta(n)$$

We assume that the system and its inverse are causal. Then this equation simplifies to

$$\sum_{k=0}^{n} h(k) h_I(n-k) = \delta(n)$$

For $n = 0$, $h_I(0) = 1/h(0)$.

For $n \geq 1$, $h_I(n)$ can be obtained recursively

$$h_I(n) = -\sum_{k=1}^{n} \frac{h(k) h_I(n-k)}{h(0)}, \quad n \geq 1$$

Minimum-Phase, Maximum-Phase, and Mixed-Phase Systems

e.g.,

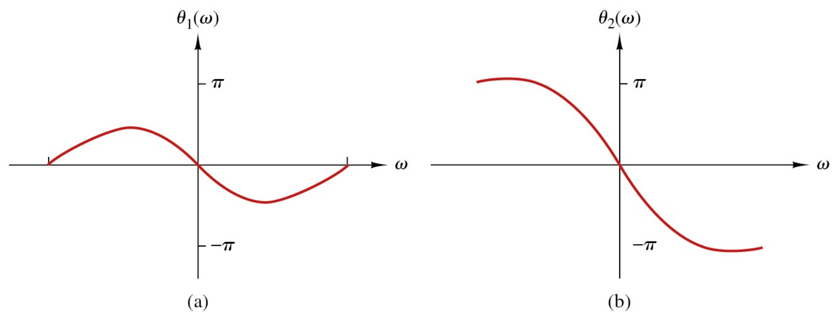
$$\begin{aligned} H_1(z) &= 1 + \frac{1}{2} z^{-1} \\ H_2(z) &= \frac{1}{2} + z^{-1} \end{aligned}$$

[Q: What are $h_1(n)$ and $h_2(n)$? A: $h_1(n) = \delta(n) + \frac{1}{2}\delta(n-1)$]

$$\begin{aligned} |H_1(\omega)| &= |H_2(\omega)| = \sqrt{\frac{5}{4} + \cos\omega} \\ \angle H_1(\omega) &= -\omega + \tan^{-1} \frac{\sin\omega}{0.5 + \cos\omega} \\ \angle H_2(\omega) &= -\omega + \tan^{-1} \frac{\sin\omega}{2 + \cos\omega} \end{aligned}$$

Minimum-Phase, Maximum-Phase, and Mixed-Phase Systems

$$\begin{aligned}\angle H_1(\omega) &= -\omega + \tan^{-1}\frac{\sin\omega}{0.5+\cos\omega}\\ \angle H_2(\omega) &= -\omega + \tan^{-1}\frac{\sin\omega}{2+\cos\omega}\end{aligned}$$



Min-phase: $\angle H(\pi) - \angle H(0) = 0$; Max-phase: $\angle H(\pi) - \angle H(0) = \pi$.
Note: For real-valued impulse responses, $\angle H(e^{j0}) = \angle H(0) = 0$.

Minimum-Phase, Maximum-Phase Systems

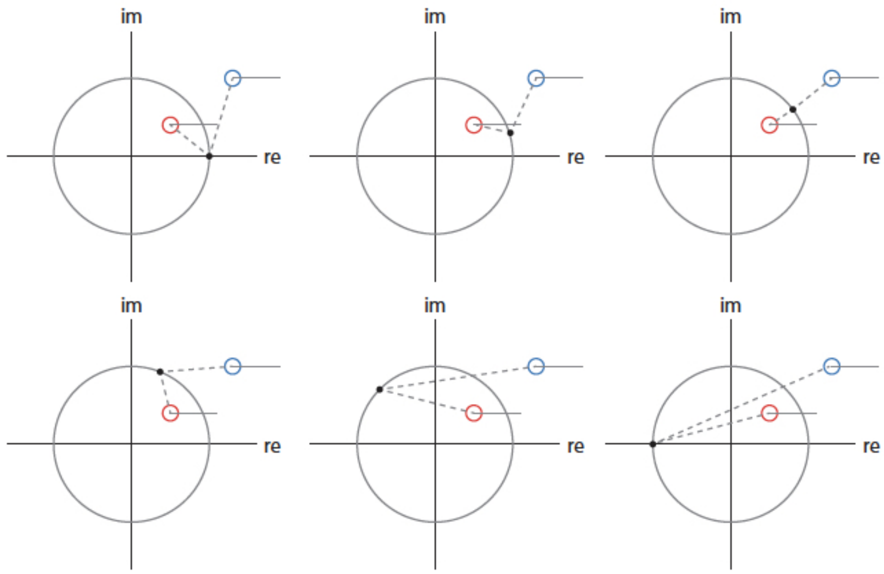
$$H(z) = \frac{b_0 \prod_{m=1}^{M}(1 - z_m z^{-1})}{a_0 \prod_{n=1}^{N}(1 - p_n z^{-1})}$$

The phase response of a rational $H(z)$ can be written as

$$\begin{aligned}\angle H(\omega) &= \angle\frac{b_0}{a_0} + \sum_{m=1}^{M}\angle(1 - z_m e^{-j\omega}) - \sum_{n=1}^{N}\angle(1 - p_n e^{-j\omega})\\ &= \angle\frac{b_0}{a_0} + \sum_{m=1}^{M}\left[\angle(e^{j\omega} - z_m) - \angle e^{j\omega}\right] - \sum_{n=1}^{N}\angle(1 - p_n e^{-j\omega})\end{aligned}$$

Minimum-Phase, Maximum-Phase Systems

Illustrate graphically: (Better explained if zeros are real.)



Minimum-Phase, Maximum-Phase, and Mixed-Phase Systems

For an FIR system that has M zeros,

$$H(\omega) = b_0(1 - z_1 e^{-j\omega})(1 - z_2 e^{-j\omega}) \cdots (1 - z_M e^{-j\omega})$$

- When all zeros are inside the unit circle, Minimum-phase: $\angle H(\pi) - \angle H(0) = 0$;
- When all zeros are outside the unit circle, Maximum-phase: $\angle H(\pi) - \angle H(0) = M\pi$.

If the FIR system with M zeros has some of its zeros inside the unit circle and the remaining zeros outside the unit circle, it is called a mixed-phase system or a nonminimum-phase system.

Minimum-Phase, Maximum-Phase Systems

- A system is called a minimum-phase system if it has the minimum group delay of the set of systems that have the same magnitude response.
- A system is called a maximum-phase system if it has the maximum group delay of the set of systems that have the same magnitude response.

Minimum-Phase, Maximum-Phase Systems

A zero $a = |a|e^{j\theta_a}$ contributes the factor $1 - az^{-1}$ to the transfer function $H(z)$. Its phase contribution is

$$\begin{aligned}\phi_a(\omega) &= \angle(1 - |a|e^{-j(\omega-\theta_a)}) \\ &= \angle(1 - |a|\cos(\omega-\theta_a) + j|a|\sin(\omega-\theta_a)) \\ &= \arctan\left(\frac{|a|\sin(\omega-\theta_a)}{1-|a|\cos(\omega-\theta_a)}\right)\end{aligned}$$

It follows that the contribution to the group delay is

$$\tau_g(\omega) = -\frac{\partial}{\partial\omega}\angle(1 - ae^{-j\omega}) = \cdots = \frac{|a| - \cos(\omega-\theta_a)}{|a|^{-1} + |a| - 2\cos(\omega-\theta_a)}$$

If $|a| < 1$, the numerator gets larger if we replace $|a|$ with $|a|^{-1}$.

Minimum-Phase, Maximum-Phase, and Mixed-Phase Systems

Since the derivative of the phase characteristic of the system is a measure of the time delay that signal frequency components undergo in passing through the system,

- a minimum-phase characteristic implies a minimum delay function;
- a maximum-phase characteristic implies that the delay characteristic is also maximum.

Because

$$|H(\omega)|^2 = H(z)H(z^{-1})|_{z=e^{j\omega}},$$

if we replace a zero z_k of the system by its inverse $1/z_k$, the magnitude characteristic of the system does not change.

Place zeros inside unit circle for minimum phase.

Minimum Phase in Time-Domain

For all *causal* and *stable* systems that have the same magnitude response, the minimum phase system has its energy concentrated near the start of the impulse response $h(n)$. i.e., it minimizes the following function which we can think of as the delay of energy in the impulse response.

$$\sum_{n=m}^{\infty} |h(n)|^2, \ \ \forall m \in \mathbb{Z}^+$$

- In the set of equal-magnitude-response systems, the minimum phase system has minimum energy delay.
- In the set of equal-magnitude-response systems, the maximum phase system has maximum energy delay.

Minimum-Phase, Maximum-Phase, and Mixed-Phase Systems

Extend to IIR systems that have rational system functions

$$H(z) = \frac{B(z)}{A(z)}$$

It is minimum-phase, if all its poles and zeros are inside the unit circle.

For a stable and causal system, the system is maximum phase if all the zeros are outside the unit circle.

- A stable pole-zero system that is minimum phase has a stable inverse which is also minimum phase. Why?
- Maximum-phase systems and mixed-phase systems result in unstable inverse systems.

Decomposition of Nonminimum-Phase Pole-Zero Systems

Any nonminimum-phase pole-zero system can be expressed as

$$H(z) = H_{min}(z)H_{ap}(z)$$

$H(z)$ is causal and stable.

$B(z) = B_1(z)B_2(z)$, where $B_1(z)$ has all its roots inside the unit circle, $B_2(z)$ has all its roots outside the unit circle.

Then,

$$\begin{aligned} H_{min}(z) &= \frac{B_1(z)B_2(z^{-1})}{A(z)} \\ H_{ap}(z) &= \frac{B_2(z)}{B_2(z^{-1})} \end{aligned}$$

$H_{ap}(z)$ is a stable, all-pass, maximum-phase system.

Group delay: $\tau_g(\omega) = \tau_g^{min}(\omega) + \tau_g^{ap}(\omega)$

System Identification and Deconvolution

$$y(n) = h(n) \otimes x(n)$$

$$H(z) = \frac{Y(z)}{X(z)}$$

The system can be identified uniquely if it is known causal.

Alternatively, if the system is causal,

$$y(n) = \sum_{k=0}^{n} h(k)x(n-k), \;\; n \geq 0$$

hence, recursively, we have

$$\begin{aligned} h(0) &= \frac{y(0)}{x(0)} \\ h(n) &= \frac{y(n) - \sum_{k=0}^{n-1} h(k)x(n-k)}{x(0)}, \;\; n \geq 1 \end{aligned}$$

System Identification and Deconvolution

The crosscorrelation method is an effective and practical method for system identification.

$$r_{yx}(m) = \sum_{k=0}^{\infty} h(k) r_{xx}(m-k) = h(m) \otimes r_{xx}(m)$$

$$S_{yx}(\omega) = H(\omega) S_{xx}(\omega) = H(\omega)|X(\omega)|^2$$

Therefore,

$$H(\omega) = \frac{S_{yx}(\omega)}{S_{xx}(\omega)} = \frac{S_{yx}(\omega)}{|X(\omega)|^2}$$