ELC 4351: Digital Signal Processing

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Difference Equations

Discrete-time Systems Described by Difference Equations

A LTI system is characterized by its unit sample response $h(n)$.

The output $y(n)$ of the system for any given input $x(n)$ is determined by

$$y(n) = \sum_{k=-\infty}^{\infty} h(k)x(n-k)$$

$$y(n) = h(n) \otimes x(n)$$

FIR systems vs. IIR systems

Recursive and Nonrecursive Discrete-time Systems

e.g., Cumulative average of signal $x(n)$

$$y(n) = \frac{1}{n+1}\sum_{k=0}^{n} x(k)$$

$$\begin{aligned}(n+1)y(n) &= \sum_{k=0}^{n-1} x(k) + x(n) \\ &= ny(n-1) + x(n)\end{aligned}$$

$$y(n) = \frac{n}{n+1}y(n-1) + \frac{1}{n+1}x(n)$$

where $y(n_0 - 1)$ is the initial condition for the system at time $n = n_0$.

LTI Systems Characterized by Constant-Coefficient Difference Equations

A recursive system:

$$y(n) = \alpha y(n-1) + x(n)$$

where α is a constant.

$$\begin{aligned}y(0) &= \alpha y(-1) + x(0) \\ y(1) &= \alpha y(0) + x(1) = \alpha^2 y(-1) + \alpha x(0) + x(1) \\ \vdots \quad & \quad \vdots \\ y(n) &= \alpha^{n+1} y(-1) + \sum_{k=0}^{n} \alpha^k x(n-k), \quad n \geq 0\end{aligned}$$

LTI Systems Characterized by Constant-Coefficient Difference Equations

$$y(n) = \alpha^{n+1} y(-1) + \sum_{k=0}^{n} \alpha^k x(n-k), \;\; n \geq 0$$

1. The system is initially relaxed at time $n = 0$, i.e., $y(-1) = 0$.
   Zero-state response or forced response

   $$y_{zs}(n) = \sum_{k=0}^{n} \alpha^k x(n-k), \;\; n \geq 0$$

2. The input $x(n) = 0, \forall n$.
   Zero-input response or natural response

   $$y_{zi}(n) = \alpha^{n+1} y(-1), \;\; n \geq 0$$

3. The total response of the system

   $$y(n) = y_{zs}(n) + y_{zi}(n)$$

LTI Systems Characterized by Constant-Coefficient Difference Equations

For LTI systems, a general form of the input-output relationship.

$$y(n) = -\sum_{k=1}^{N} a_k y(n-k) + \sum_{k=0}^{M} b_k x(n-k)$$

$$\sum_{k=0}^{N} a_k y(n-k) = \sum_{k=0}^{M} b_k x(n-k), \;\; a_0 \equiv 1$$

The integer N is the order of the difference equation or the order of the system.

Linear Systems Requirements

A Linear System:

- The total response is equal to the sum of the zero-state and zero-input responses

$$y(n) = y_{zs}(n) + y_{zi}(n)$$

- The principle of superposition applies to the zero-state response.
- The principle of superposition applies to the zero-input response.

Solution of Linear Constant-Coefficient Difference Equations

The direct solution

$$y(n) = \underbrace{y_h(n)}_{\text{homogeneous solution}} + \underbrace{y_p(n)}_{\text{particular solution}}$$

The Homogeneous Solution of A Difference Equation

The homogeneous difference equation:

$$\sum_{k=0}^{N} \alpha_k y(n-k) = 0$$

- We assume that the solution is in the form of an exponential, i.e., $y_h(n) = \lambda^n$.
- Substituting this in the equation, we obtain the polynomial equation

$$\sum_{k=0}^{N} \alpha_k \lambda^{n-k} = 0$$

$$\lambda^{n-N} \underbrace{(\lambda^N + \alpha_1 \lambda^{N-1} + \cdots + \alpha_{N-1}\lambda + \alpha_N)}_{\text{characteristic polynomial}} = 0$$

- The characteristic polynomial of the system has N roots: $\lambda_1, \lambda_2, \ldots, \lambda_N$.

The Homogeneous Solution of A Difference Equation

If the N roots are distinct, the general solution to the homogeneous difference equation is

$$y_h(n) = C_1\lambda_1^n + C_2\lambda_2^n + \cdots + C_N\lambda_N^n$$

where $C_1, C_2, \ldots, C_N$ are weighting coefficients.

These coefficients are determined from the initial conditions of the system.

$y_h(n)$ is the zero-input response of the system.

The Homogeneous Solution of A Difference Equation

If the characteristic polynomial contains multiple roots, e.g., λ_1 is a root of multiplicity m, then

$$y_h(n) = C_1\lambda_1^n + C_2 n\lambda_1^n + \cdots + C_m n^{m-1}\lambda_1^n + C_{m+1}\lambda_{m+1}^n + \cdots + C_N\lambda_M^n$$

The Particular Solution of A Difference Equation

The particular difference equation for a specific input signal $x(n)$:

$$\sum_{k=0}^{N} a_k y_p(n-k) = \sum_{k=0}^{M} b_k x(n-k), \;\; a_0 \equiv 1$$

Input Signal $x(n)$	Particular Solution $y_p(n)$
A	K
AM^n	KM^n
An^M	$K_0 n^M + K_1 n^{M-1} + \cdots + K_M$
$A^n n^M$	$A^n(K_0 n^M + K_1 n^{M-1} + \cdots + K_M)$
$A\cos\omega_0 n$	$K_1\cos\omega_0 n + K_2\sin\omega_0 n$
$A\sin\omega_0 n$	$K_1\cos\omega_0 n + K_2\sin\omega_0 n$

The Total Solution of A Difference Equation

$$y(n) = y_h(n) + y_p(n)$$

The Impulse Response of a LTI Recursive System

- The impulse response $h(n)$ is equal to the zero-state response of the system (the system is initially relaxed) when the input $x(n) = \delta(n)$.

  $$y_{zs}(n) = \sum_{k=0}^{n} h(k)x(n-k), \quad n \geq 0$$

  When $x(n) = \delta(n)$, $y_{zs}(n) = h(n)$.

- If the excitation is an impulse, the particular solution is zero, since $x(n) = 0, \forall n > 0$. That is $y_p(n) = 0$.

  The response of the system to an impulse consists only of the solution to the homogeneous equations.

The Impulse Response of a LTI Recursive System

Nth-order linear difference equation.

The solution of the homogeneous equation is

$$y_h(n) = \sum_{k=1}^{N} C_k \lambda_k^n.$$

Hence, the impulse response of the system is

$$h(n) = \sum_{k=1}^{N} C_k \lambda_k^n.$$